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Construction of the Energy-Momentum Tensor for Wilson Actions

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Abstract

Given an arbitrary Wilson action of a real scalar field, we discuss how to construct the energy-momentum tensor of the theory. Using the exact renormalization group, we can determine the energy-momentum tensor implicitly, but we are short of obtaining an explicit formula in terms of the Wilson action.

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I. INTRODUCTION

The energy-momentum tensor is important for quantum field theory; it describes the conservation of energy, momentum, and angular momentum, and it also determines how the theory couples to weak external gravity. Long ago the construction of a symmetric energy-momentum tensor was discussed extensively in the canonical formalism by Belinfante [1] and Rosenfeld [2]. It was shown that, given a lagrangian, an explicit formula can be given for the energy-momentum tensor. More recently (but also long ago) the construction of the "improved" energy-momentum tensor was discussed in [3], where renormalizability of the energy-momentum tensor in renormalized theories was proven within perturbation theory.

In this paper we are interested in constructing the energy-momentum tensor of a generic real scalar theory, given its Wilson action. A Wilson action incorporates a momentum cutoff so that its functional integrals are well defined. But it contains all possible local terms, and no canonical method can be adopted for the construction of the energy-momentum tensor. The most straightforward approach would be to couple the theory to external gravity, and define the energy-momentum tensor as the functional derivative of the action with respect to the external metric. However, this adds further complexity to the study, and we would like to remain in flat space if possible.

In constructing the energy-momentum tensor, we need an approach suitable to Wilson actions. Local composite operators, introduced in [4] (sect. 12.4 and Appendix) and [5], are infinitesimal deformations of a Wilson action, and the energy-momentum tensor is a particular example. We will formulate the Ward identity for the energy-momentum tensor using composite operators, in particular, equation-of-motion composite operators.[6] We will see that the equation-of-motion composite operators give us sufficient information to determine the symmetric energy-momentum tensor. We are short of obtaining an explicit formula in terms of the action, but we are left with only the familiar ambiguity that would correspond to the coupling to the Riemann curvature tensor if external gravity were present.[7]

Throughout the paper, we resort to the assumption of locality of composite operators: if the space integral of a local composite operator vanishes, the operator must be a derivative. In the Fourier space, if a local composite operator vanishes at zero momentum, it must be proportional to a momentum:

$$\mathcal{O}(p=0)=0 \Longrightarrow \mathcal{O}(p)=p_\mu J_\mu(p) \tag{1}$$

where $J_\mu(p)$ is also a local composite operator.

Our present work was motivated by a preliminary presentation of the work [8] in which the conformal invariance [9] of the $O(N)$ invariant scalar theory at its fixed point was proven using Wilson actions and the exact renormalization group. In a recent work by Rosten [10] the energy-momentum tensor has been constructed for scale invariant Wilson actions. Our section VI has some overlap with part of the results obtained in [10].

This paper is organized as follows. In sect. II we briefly review Wilson actions and the exact renormalization group (ERG) transformation following the results of [11]. Additional details are given in Appendices A & B. In sect. III we derive the operator equations starting from the naive invariance of the Wilson action under translations and rotations. In sect. IV we derive the Ward identity for translation invariance that determines a symmetric energy-momentum tensor implicitly. In sect. V we use ERG to constrain the trace of the energy-momentum tensor. In sect. VI we discuss the energy-momentum tensor for a fixed point Wilson action. Under the assumption, formulated in [7], we derive the Ward identity for conformal invariance. In sect. VII we show how our construction works using the massless free theory as an example. We conclude the paper in sect. VIII.

Throughout the paper we work in D-dimensional Euclidean momentum space, and we use the abbreviated notation

$$\int_p = \int \frac{d^D p}{(2\pi)^D}, \quad \delta(p) = (2\pi)^D \delta^{(D)}(p) \tag{2}$$

II. WILSON ACTIONS

We consider a real scalar theory in D-dimensional Euclidean space. We denote the Fourier transform of the scalar field by $\phi(p)$. Let $S[\phi]$ be a Wilson action that incorporates a momentum cutoff so that the correlation functions defined by

$$\langle \phi(p_1) \cdots \phi(p_n) \rangle_S \equiv \int [d\phi]\, \phi(p_1) \cdots \phi(p_n)\, \mathrm{e}^{S[\phi]} \tag{3}$$

are free from ultraviolet divergences. We measure all dimensionful quantities in units of an appropriate power of the cutoff and deal only with dimensionless quantities. Hence, our momentum cutoff becomes 1 in this convention.

We introduce a positive cutoff function $K(p)$ that depends on the squared momentum

$p^2 = p_\mu p_\mu$ and decreases rapidly for $p^2 > 1$. We impose

$$K(p) \stackrel{p^2\to 0}{\longrightarrow} 1 \tag{4}$$

so that the cutoff does not affect low energy physics. Using $K(p)$, we define modified correlation functions by

$$\begin{aligned}\langle\!\langle \phi(p_1)\cdots\phi(p_n)\rangle\!\rangle_S &\equiv \prod_{i=1}^n \frac{1}{K(p_i)}\\ &\times \left\langle \exp\left(-\int_p \frac{K(p)\left(1-K(p)\right)}{p^2}\frac{1}{2}\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}\right)\phi(p_1)\cdots\phi(p_n)\right\rangle_S\end{aligned} \tag{5}$$

The exact renormalization group (ERG) transformation R_t acting on S is defined so that the modified correlation functions are simply related as

$$\left\langle\!\!\left\langle \phi(p_1\mathrm{e}^t)\cdots\phi(p_n\mathrm{e}^t)\right\rangle\!\!\right\rangle_{R_t S} = \mathrm{e}^{t\cdot n\left(-\frac{D+2}{2}+\gamma\right)}\langle\!\langle \phi(p_1)\cdots\phi(p_n)\rangle\!\rangle_S \tag{6}$$

where the anomalous dimension γ of ϕ is chosen so that R_t has a fixed point.[11]

Besides the Wilson action, local composite operators play a very important role in this paper. A local composite operator $\mathcal{O}(p)$ is a functional of ϕ which can be interpreted as an infinitesimal deformation of the Wilson action.[4, 5] We define its modified correlation functions by

$$\begin{aligned}\langle\!\langle \mathcal{O}(p)\,\phi(p_1)\cdots\phi(p_n)\rangle\!\rangle_S &\equiv \prod_{i=1}^n \frac{1}{K(p_i)}\\ &\times \left\langle \mathcal{O}(p)\exp\left(-\int_q \frac{K(q)\left(1-K(q)\right)}{q^2}\frac{1}{2}\frac{\delta^2}{\delta\phi(q)\delta\phi(-q)}\right)\phi(p_1)\cdots\phi(p_n)\right\rangle_S\end{aligned} \tag{7}$$

ERG acts on $\mathcal{O}$ such that

$$\left\langle\!\!\left\langle \left(R_t\mathcal{O}\right)(p\mathrm{e}^t)\,\phi(p_1\mathrm{e}^t)\cdots\phi(p_n\mathrm{e}^t)\right\rangle\!\!\right\rangle_{R_t S} = \mathrm{e}^{tn\left(-\frac{D+2}{2}+\gamma\right)}\langle\!\langle \mathcal{O}(p)\,\phi(p_1)\cdots\phi(p_n)\rangle\!\rangle_S \tag{8}$$

(See Appendices A and B for more details.)

III. INVARIANCE UNDER TRANSLATIONS AND ROTATIONS

We assume that the theory described by the Wilson action $S[\phi]$ is invariant under both translations and rotations. Let us derive the consequences of this assumption. Some relevant technical details on composite operators are given in Appendix A.

A. Translation Invariance

Translation invariance implies that the correlation functions satisfy the Ward identity

$$\sum_{i=1}^{n} p_{i\mu} \langle\!\langle \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_S = 0 \tag{9}$$

This is equivalent to the operator identity

$$\int_q K(q) \mathrm{e}^{-S} \frac{\delta}{\delta\phi(q)} \left(q_\mu \left[\phi(q)\right] \mathrm{e}^{S} \right) = 0 \tag{10}$$

where

$$[\phi(q)] \equiv \frac{1}{K(q)} \left(\phi(q) + \frac{K(q)\,(1-K(q))}{q^2} \frac{\delta S}{\delta\phi(-q)} \right) \tag{11}$$

is the composite operator satisfying

$$\langle\!\langle [\phi(q)]\, \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_S = \langle\!\langle \phi(q)\phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_S \tag{12}$$

In the following we wish to show, in some details, that the operator identity (10) is a consequence of the naive translation invariance

$$\int_q q_\mu \phi(q) \frac{\delta S}{\delta\phi(q)} = 0 \tag{13}$$

which can be rewritten as

$$\int d^D r\, \partial_\mu \phi(r) \frac{\delta S}{\delta\phi(r)} = 0 \tag{14}$$

in coordinate space. The derivation is straightforward. Defining

$$k(q) \equiv K(q)\,(1-K(q)) \tag{15}$$

for simplicity, we find

$$\begin{aligned}
&\int_q K(q) \mathrm{e}^{-S} \frac{\delta}{\delta\phi(q)} \left(q_\mu \left[\phi(q)\right] \mathrm{e}^{S} \right) \\
&= \int_q \mathrm{e}^{-S} \frac{\delta}{\delta\phi(q)} \left[q_\mu \left(\phi(q) + \frac{k(q)}{q^2} \frac{\delta S}{\delta\phi(-q)} \right) \mathrm{e}^{S} \right] \\
&= \int_q \left[q_\mu \delta(0) + q_\mu \frac{k(q)}{q^2} \frac{\delta^2 S}{\delta\phi(q)\delta\phi(-q)} + q_\mu \left(\phi(q) + \frac{k(q)}{q^2} \frac{\delta S}{\delta\phi(-q)} \right) \frac{\delta S}{\delta\phi(q)} \right] \\
&= \int_q q_\mu \phi(q) \frac{\delta S}{\delta\phi(q)} + \int_q q_\mu \frac{k(q)}{q^2} \left\{ \frac{\delta^2 S}{\delta\phi(q)\delta\phi(-q)} + \frac{\delta S}{\delta\phi(-q)} \frac{\delta S}{\delta\phi(q)} \right\}
\end{aligned} \tag{16}$$

The first term vanishes due to (13), and the second term vanishes since the integrand is odd under the inversion $q \to -q$. Hence, we obtain (10).

B. Rotation Invariance

Rotation invariance implies the Ward identity

$$\epsilon_{\mu\nu} \sum_{i=1}^{n} p_{i\nu} \left\langle\!\!\left\langle \phi(p_1) \cdots \frac{\partial}{\partial p_{i\mu}} \phi(p_i) \cdots \phi(p_n) \right\rangle\!\!\right\rangle_S = 0 \tag{17}$$

for an arbitrary constant antisymmetric tensor $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$. This is equivalent to the operator identity

$$\epsilon_{\mu\nu} \int_q K(q) \mathrm{e}^{-S} \frac{\delta}{\delta \phi(q)} \left(q_\nu \frac{\partial}{\partial q_\mu} [\phi(q)] \, \mathrm{e}^S \right) = 0 \tag{18}$$

We wish to show, in the following, that this operator identity results from the naive rotation invariance of the action:

$$\int_q \epsilon_{\mu\nu} q_\nu \frac{\partial \phi(q)}{\partial q_\mu} \frac{\delta S}{\delta \phi(q)} = 0 \tag{19}$$

The derivation is again straightforward.

Using the rotation invariance of $k(q) = K(q) \, (1 - K(q))$, we obtain

$$\begin{aligned}
&\int_q K(q) \mathrm{e}^{-S} \epsilon_{\mu\nu} q_\nu \frac{\delta}{\delta \phi(q)} \left(\frac{\partial}{\partial q_\mu} [\phi(q)] \ \mathrm{e}^S \right) \\
&= \int_q \mathrm{e}^{-S} \frac{\delta}{\delta \phi(q)} \left[\epsilon_{\mu\nu} q_\nu \frac{\partial}{\partial q_\mu} \left(\phi(q) + \frac{k(q)}{q^2} \frac{\delta S}{\delta \phi(-q)} \right) \mathrm{e}^S \right] \\
&= \int_q \frac{k(q)}{q^2} \left(\epsilon_{\mu\nu} q_\nu \frac{\partial}{\partial q_\mu} \frac{\delta^2 S}{\delta \phi(-q) \delta \phi(q')} \right)_{q' \to q} \\
&\quad + \int_q \epsilon_{\mu\nu} q_\nu \frac{\partial \phi(q)}{\partial q_\mu} \frac{\delta S}{\delta \phi(q)} \mathrm{e}^S + \int_q \frac{k(q)}{q^2} \epsilon_{\mu\nu} q_\nu \frac{\partial}{\partial q_\mu} \frac{\delta S}{\delta \phi(-q)} \cdot \frac{\delta S}{\delta \phi(q)}
\end{aligned} \tag{20}$$

The second term on the right vanishes due to (19). Hence, we obtain

$$\begin{aligned}
&\int_q K(q) \mathrm{e}^{-S} \epsilon_{\mu\nu} q_\nu \frac{\delta}{\delta \phi(q)} \left(\frac{\partial}{\partial q_\mu} [\phi(q)] \ \mathrm{e}^S \right) \\
&= \int_q \frac{k(q)}{q^2} \frac{1}{2} \epsilon_{\mu\nu} q_\nu \frac{\partial}{\partial q_\mu} \left(\frac{\delta^2 S}{\delta \phi(-q) \delta \phi(q)} + \frac{\delta S}{\delta \phi(-q)} \frac{\delta S}{\delta \phi(q)} \right) \\
&= - \int_q \epsilon_{\mu\nu} q_\nu \frac{\partial}{\partial q_\mu} \frac{k(q)}{q^2} \cdot \frac{1}{2} \left(\frac{\delta^2 S}{\delta \phi(-q) \delta \phi(q)} + \frac{\delta S}{\delta \phi(-q)} \frac{\delta S}{\delta \phi(q)} \right)
\end{aligned} \tag{21}$$

This vanishes thanks to the rotation invariance of $k(q)$. Hence, we obtain (18).

IV. CONSTRUCTION OF THE ENERGY-MOMENTUM TENSOR

Given a Wilson action $S[\phi]$ with the invariance under translations and rotations, we would like to construct the energy-momentum tensor.

To start with, we introduce a local composite operator with momentum p, defined by

$$J_\mu(p) \equiv \int_q K(q) \mathrm{e}^{-S} \frac{\delta}{\delta\phi(q)} \left((p+q)_\mu \left[\phi(p+q)\right] \mathrm{e}^{S} \right) \tag{22}$$

This vanishes at $p=0$ as a consequence of the translation invariance (10):

$$J_\mu(0) = 0 \tag{23}$$

Since $J_\mu(p)$ is a local composite operator that vanishes at $p=0$, it must be proportional to p so that

$$J_\mu(p) = p_\nu \Theta_{\nu\mu}(p) \tag{24}$$

in terms of another local composite operator $\Theta_{\nu\mu}(p)$, which we call the energy-momentum tensor. Note that this relation does not determine $\Theta_{\nu\mu}(p)$ uniquely; it has an additive ambiguity by

$$p_\alpha Y_{\nu\alpha,\mu}(p) \tag{25}$$

where

$$Y_{\alpha\nu,\mu} = -Y_{\nu\alpha,\mu} \tag{26}$$

We now differentiate (24) with respect to p_α to obtain

$$\Theta_{\alpha\mu}(p) + p_\nu \frac{\partial}{\partial p_\alpha} \Theta_{\nu\mu}(p) = \int_q K(q) \mathrm{e}^{-S} \frac{\delta}{\delta\phi(q)} \left[\left\{ \delta_{\alpha\mu} + (p+q)_\mu \frac{\partial}{\partial q_\alpha} \right\} \left[\phi(p+q)\right] \mathrm{e}^{S} \right] \tag{27}$$

Antisymmetrizing this with respect to α and μ, we obtain

$$\begin{aligned} &\Theta_{\alpha\mu}(p) - \Theta_{\mu\alpha}(p) + p_\nu \left(\frac{\partial \Theta_{\nu\mu}(p)}{\partial p_\alpha} - \frac{\partial \Theta_{\nu\alpha}(p)}{\partial p_\mu} \right) \\ &= \int_q K(q) \mathrm{e}^{-S} \frac{\delta}{\delta\phi(q)} \left[\left\{ (q+p)_\mu \frac{\partial}{\partial q_\alpha} - (q+p)_\alpha \frac{\partial}{\partial q_\mu} \right\} \left[\phi(p+q)\right] \mathrm{e}^{S} \right] \end{aligned} \tag{28}$$

Rotation invariance implies the vanishing of the right-hand side in the limit $p=0$. Hence, we find

$$\Theta_{\alpha\mu}(p) - \Theta_{\mu\alpha}(p) \stackrel{p\to 0}{\longrightarrow} 0 \tag{29}$$

Since $\Theta_{\mu\nu}(p)$ is a local composite operator, this implies that the antisymmetric part of $\Theta_{\mu\nu}(p)$ is a local composite operator proportional to p:

$$\Theta_{\mu\nu}(p) - \Theta_{\nu\mu}(p) = p_\alpha \tau_{\alpha,\mu\nu}(p) \tag{30}$$

where

$$\tau_{\alpha,\mu\nu}(p) = -\tau_{\alpha,\nu\mu}(p) \tag{31}$$

Using $\tau_{\alpha,\mu\nu}$ we can construct a symmetric energy-momentum tensor following the procedure given in [1] and [2]. We define

$$B_{\alpha\mu\nu}(p) \equiv \frac{1}{2}\left(\tau_{\alpha,\mu\nu}(p) + \tau_{\mu,\nu\alpha}(p) - \tau_{\nu,\alpha\mu}(p)\right) \tag{32}$$

which is antisymmetric with respect to the first two indices

$$B_{\alpha\mu\nu}(p) = -B_{\mu\alpha\nu}(p) \tag{33}$$

and

$$B_{\alpha\mu\nu}(p) - B_{\alpha\nu\mu}(p) = \tau_{\alpha,\mu\nu}(p) \tag{34}$$

We then redefine the energy-momentum tensor by

$$\Theta'_{\mu\nu}(p) \equiv \Theta_{\mu\nu}(p) - p_\alpha B_{\alpha\mu\nu}(p) \tag{35}$$

which, on account of (33), satisfies

$$p_\mu \Theta_{\mu\nu}(p) = p_\mu \Theta'_{\mu\nu}(p) \tag{36}$$

and is, thanks to (34), symmetric

$$\Theta'_{\mu\nu}(p) = \Theta'_{\nu\mu}(p) \tag{37}$$

We will omit the prime on $\Theta'_{\mu\nu}$ from now on.

To summarize so far, our energy-momentum tensor $\Theta_{\mu\nu}(p)$ is symmetric, and it satisfies

$$p_\mu \Theta_{\mu\nu}(p) = \int_q K(q) \mathrm{e}^{-S} \frac{\delta}{\delta\phi(q)} \Big((p+q)_\nu \left[\phi(p+q)\right] \mathrm{e}^S \Big) \tag{38}$$

for translation invariance, and

$$\begin{aligned} & p_\nu \left(\frac{\partial \Theta_{\nu\mu}(p)}{\partial p_\alpha} - \frac{\partial \Theta_{\nu\alpha}(p)}{\partial p_\mu} \right) \\ & = \int_q K(q) \mathrm{e}^{-S} \frac{\delta}{\delta\phi(q)} \left(\left\{ (q+p)_\mu \frac{\partial}{\partial q_\alpha} - (q+p)_\alpha \frac{\partial}{\partial q_\mu} \right\} \left[\phi(p+q)\right] \mathrm{e}^S \right) \end{aligned} \tag{39}$$

for rotation invariance. (39) is a consequence of the symmetry $\Theta_{\mu\nu} = \Theta_{\nu\mu}$ and (38). The energy-momentum tensor $\Theta_{\mu\nu}(p)$ is now left with an additive ambiguity by a symmetric tensor

$$p_\alpha p_\beta Y_{\mu\alpha,\nu\beta}(p) \tag{40}$$

where $Y_{\mu\alpha,\nu\beta}(p)$ must satisfy

$$p_\mu p_\alpha p_\beta Y_{\mu\alpha,\nu\beta}(p) = 0 \tag{41}$$

This implies [7]

$$Y_{\mu\alpha,\nu\beta} = Y_{\nu\beta,\mu\alpha} = -Y_{\alpha\mu,\nu\beta} = -Y_{\mu\alpha,\beta\nu} \tag{42}$$

which is the symmetry of the Riemann curvature tensor $R_{\mu\alpha\nu\beta}$.

V. CONSISTENCY WITH ERG

We now combine the results of the previous section with ERG. Differentiating (38) with respect to p_ν and summing over ν, we obtain

$$\begin{aligned}
&\Theta(p) + p_\mu \frac{\partial}{\partial p_\nu} \Theta_{\mu\nu}(p) \\
&= -D\mathcal{N}(p) + \int_q K(q) \mathrm{e}^{-S} \frac{\delta}{\delta\phi(q)} \left[(q+p)_\nu \frac{\partial}{\partial q_\nu} \left[\phi(p+q)\right] \mathrm{e}^S \right]
\end{aligned} \tag{43}$$

where $\Theta \equiv \Theta_{\mu\mu}$ is the trace, and

$$\mathcal{N}(p) \equiv -\int_q K(q) \mathrm{e}^{-S} \frac{\delta}{\delta\phi(q)} \left(\left[\phi(q+p)\right] \mathrm{e}^S \right) \tag{44}$$

is an equation-of-motion composite operator satisfying

$$\langle\!\langle \mathcal{N}(p)\phi(p_1)\cdots\phi(p_n)\rangle\!\rangle_S = \sum_{i=1}^n \langle\!\langle \phi(p_1)\cdots\phi(p_i+p)\cdots\phi(p_n)\rangle\!\rangle_S \tag{45}$$

In the limit $p \to 0$, (43) gives

$$\Theta(0) = -D\mathcal{N} + \int_q K(q)\mathrm{e}^{-S} \frac{\delta}{\delta\phi(q)} \left[q_\mu \frac{\partial}{\partial q_\mu} \left[\phi(q)\right] \mathrm{e}^S \right] \tag{46}$$

where we denote $\mathcal{N} \equiv \mathcal{N}(p=0)$.

The trace $\Theta(0)$ is related to the ERG differential equation, which is briefly reviewed in Appendix B. ERG acts on the Wilson action S as

$$\mathcal{D}S[\phi] \equiv \frac{d}{dt}\left(R_t S\right)\Big|_{t=0} = \int_q K(q) \mathrm{e}^{-S[\phi]} \frac{\delta}{\delta\phi(q)} \left[q_\mu \frac{\partial}{\partial q_\mu} \left[\phi(q)\right] \mathrm{e}^{S[\phi]} \right] + \left(-\frac{D+2}{2} + \gamma \right) \mathcal{N} \tag{47}$$

(This is (B5) of Appendix B.) Comparing this with (46), we obtain

$$\Theta(0) = \mathcal{D}S - \left(\frac{D-2}{2} + \gamma \right) \mathcal{N} \tag{48}$$

where $\frac{D-2}{2} + \gamma$ is the full scale dimension of the scalar field in coordinate space. Generalizing this to non-vanishing momenta, we obtain

$$\Theta(p) = \mathcal{O}(p) - \left(\frac{D-2}{2} + \gamma\right) \mathcal{N}(p) \tag{49}$$

where $\mathcal{O}(p)$ is a local composite operator whose zero momentum limit is

$$\mathcal{O}(0) = \mathcal{D}S \tag{50}$$

defined by (47). Please recall that $\Theta_{\mu\nu}(p)$ is ambiguous by $p_\alpha p_\beta Y_{\mu\alpha,\nu\beta}(p)$. This translates into the ambiguity of $\Theta(p)$ (and also $\mathcal{O}(p)$) by $p_\alpha p_\beta Y_{\mu\alpha,\mu\beta}(p)$.

VI. THE ENERGY-MOMENTUM TENSOR AT A FIXED POINT

Let us suppose S is a fixed point Wilson action S^* corresponding to the anomalous dimension γ for the scalar field. We then find

$$\mathcal{O}(0) = 0 \tag{51}$$

so that there must be a local composite operator $K_\mu(p)$ that gives

$$\mathcal{O}(p) = p_\mu K_\mu(p) \tag{52}$$

Hence, (49) becomes

$$\Theta(p) = p_\mu K_\mu(p) - \left(\frac{D-2}{2} + \gamma\right) \mathcal{N}(p) \tag{53}$$

Especially, at $p = 0$, we obtain

$$\Theta(0) = -\left(\frac{D-2}{2} + \gamma\right) \mathcal{N} \tag{54}$$

which has been obtained also in [10] (see (4.44) at the end of IV D of [10]).

We now consider local composite operators at the fixed point, which form an infinite dimensional linear space. A local composite operator $\mathcal{O}$ with scale dimension $-y$ satisfies

$$R_t \mathcal{O} = \mathrm{e}^{+yt} \mathcal{O} \tag{55}$$

so that (8) gives

$$\left\langle\!\!\left\langle \mathcal{O}(p\mathrm{e}^t)\, \phi(p_1 \mathrm{e}^t) \cdots \phi(p_n \mathrm{e}^t) \right\rangle\!\!\right\rangle_{S^*} = \mathrm{e}^{t\left\{-y+n\left(-\frac{D+2}{2}+\gamma\right)\right\}} \left\langle\!\!\left\langle \mathcal{O}(p)\, \phi(p_1) \cdots \phi(p_n) \right\rangle\!\!\right\rangle_{S^*} \tag{56}$$

at the fixed point. We assume that we can choose a basis consisting of composite operators with definite scale dimensions. An arbitrary composite operator can then be given as a linear combination of various composite operators with various scale dimensions.

The translation invariance (38) implies that $\Theta_{\mu\nu}(p)$ has scale dimension 0. (In coordinate space $\Theta_{\mu\nu}(x) \equiv \int_p \mathrm{e}^{ipx}\Theta_{\mu\nu}(p)$ has scale dimension D.) Since $\mathcal{N}(p)$ also has scale dimension 0, the current $K_\mu(p)$ must have scale dimension -1. (Hence, scale dimension $D-1$ in coordinate space.) If there exists no well-defined local composite operator $K_\mu(p)$ that has scale dimension -1, i.e.,

$$\left(R_t K_\mu\right)(p) = \mathrm{e}^t K_\mu(p) \tag{57}$$

then $\mathcal{O}(p)$ inevitably vanishes, as has been also pointed out in the note added of [10]. (The non-trivial differential equation corresponding to (57) is obtained from (B12) of Appendix B with $y=1$.) For the real scalar theory in $D=3$, it has recently been shown in [8] that a well defined local composite operator $K_\mu(p)$ of scale dimension -1 does not exist so that $\mathcal{O}(p)$ vanishes. Hence, as is explained below, scale invariance implies conformal invariance for the critical Ising model.

Now, it has been of much interest lately whether scale invariance implies conformal invariance or not. (See [8, 10] within the context of ERG. See also [12] for a recent review.) As has been shown in [7], conformal invariance is equivalent to

$$\mathcal{O}(p) = p_\mu p_\nu L_{\mu\nu}(p) \tag{58}$$

where $L_{\mu\nu}(p)$ is a symmetric local composite operator of scale dimension -2. Note that the ambiguity of $\mathcal{O}(p)$ by $p_\alpha p_\beta Y_{\mu\alpha,\nu\beta}(p)$ affects $L_{\mu\nu}(p)$ by $Y_{\alpha\mu,\alpha\nu}(p)$. Hence, it does not affect the conclusion below. (In fact as has been shown in [7] we can redefine $\Theta_{\mu\nu}(p)$ to make $\mathcal{O}(p)=0$ if (58) holds.)

If (58) holds, we can derive the Ward identity for conformal invariance. In coordinate space the infinitesimal conformal transformation is given by [9]

$$\delta\phi(x) = \epsilon_\alpha \left(-x_\alpha x_\nu \partial_\nu + \frac{1}{2}x^2\partial_\alpha - \left(\frac{D-2}{2}+\gamma\right)x_\alpha\right)\phi(x) \tag{59}$$

where the last term proportional to x_α is determined by the full scale dimension of the scalar field. Going to the momentum space, we obtain

$$\delta\phi(p) = i\epsilon_\alpha \left(p_\nu \frac{\partial^2}{\partial p_\nu \partial p_\alpha} - \frac{1}{2}p_\alpha \frac{\partial^2}{\partial p_\nu \partial p_\nu} + \left(D - \left(\frac{D-2}{2}+\gamma\right)\right)\frac{\partial}{\partial p_\alpha}\right)\phi(p) \tag{60}$$

To derive the Ward identity, we use

$$\Theta(p) = p_\mu p_\nu L_{\mu\nu}(p) - \left(\frac{D-2}{2} + \gamma\right)\mathcal{N}(p) \tag{61}$$

and compute

$$\begin{aligned}
&p_\mu \left(\frac{\partial^2}{\partial p_\alpha \partial p_\nu} - \frac{1}{2}\delta_{\alpha\nu}\frac{\partial^2}{\partial p_\beta \partial p_\beta}\right)\Theta_{\nu\mu}(p)\\
&= \left(\frac{\partial^2}{\partial p_\alpha \partial p_\nu} - \frac{1}{2}\delta_{\alpha\nu}\frac{\partial^2}{\partial p_\beta \partial p_\beta}\right)(p_\mu \Theta_{\nu\mu}(p)) - \frac{\partial}{\partial p_\alpha}\Theta(p)\\
&= \left(\frac{\partial^2}{\partial p_\alpha \partial p_\nu} - \frac{1}{2}\delta_{\alpha\nu}\frac{\partial^2}{\partial p_\beta \partial p_\beta}\right)\int_q K(q)\mathrm{e}^{-S^*}\frac{\delta}{\delta\phi(q)}\left[(q+p)_\nu\left[\phi(q+p)\right]\mathrm{e}^{S^*}\right]\\
&\quad + \frac{\partial}{\partial p_\alpha}\left(\left(\frac{D-2}{2}+\gamma\right)\mathcal{N}(p) - p_\mu p_\nu L_{\mu\nu}(p)\right)
\end{aligned} \tag{62}$$

where we have used (38) and (61). We then obtain

$$\begin{aligned}
&p_\mu \left(\frac{\partial^2}{\partial p_\alpha \partial p_\nu} - \frac{1}{2}\delta_{\alpha\nu}\frac{\partial^2}{\partial p_\beta \partial p_\beta}\right)\Theta_{\nu\mu}(p)\\
&= \int_q K(q)\mathrm{e}^{-S^*}\frac{\delta}{\delta\phi(q)}\left[\left(\frac{\partial^2}{\partial p_\alpha \partial p_\nu} - \frac{1}{2}\delta_{\alpha\nu}\frac{\partial^2}{\partial p_\beta \partial p_\beta}\right)\left\{(q+p)_\nu\left[\phi(q+p)\right]\right\}\mathrm{e}^{S^*}\right.\\
&\qquad \left. - \left(\frac{D-2}{2}+\gamma\right)\frac{\partial}{\partial p_\alpha}\left[\phi(q+p)\right]\mathrm{e}^{S^*}\right] - \frac{\partial}{\partial p_\alpha}\left(p_\mu p_\nu L_{\mu\nu}(p)\right)\\
&= \int_q K(q)\mathrm{e}^{-S^*}\frac{\delta}{\delta\phi(q)}\left[\left(\left\{(q+p)_\nu\frac{\partial^2}{\partial p_\alpha \partial p_\nu} - \frac{1}{2}(q+p)_\alpha\frac{\partial^2}{\partial p_\beta \partial p_\beta}\right\}\left[\phi(q+p)\right]\right.\right.\\
&\qquad \left.\left. + \left(\frac{D+2}{2}-\gamma\right)\frac{\partial}{\partial p_\alpha}\left[\phi(p+q)\right]\right)\mathrm{e}^{S^*}\right] - \frac{\partial}{\partial p_\alpha}\left(p_\mu p_\nu L_{\mu\nu}(p)\right)
\end{aligned} \tag{63}$$

In the limit $p \to 0$ the left-hand side vanishes, and we obtain

$$\begin{aligned}
0 = \int_q K(q)\mathrm{e}^{-S^*}\frac{\delta}{\delta\phi(q)}&\left[\left\{\left(q_\nu\frac{\partial^2}{\partial q_\alpha \partial q_\nu} - \frac{1}{2}q_\alpha\frac{\partial^2}{\partial q_\beta \partial q_\beta}\right)\left[\phi(q)\right]\right.\right.\\
&\left.\left. + \left(\frac{D+2}{2}-\gamma\right)\frac{\partial}{\partial q_\alpha}\left[\phi(q)\right]\right\}\mathrm{e}^{S^*}\right]
\end{aligned} \tag{64}$$

This gives the Ward identity for conformal invariance

$$\begin{aligned}
&\sum_{i=1}^n \left(p_{i\nu}\frac{\partial^2}{\partial p_{i\alpha}\partial p_{i\nu}} - \frac{1}{2}p_{i\alpha}\frac{\partial^2}{\partial p_{i\beta}\partial p_{i\beta}} + D\frac{\partial}{\partial p_{i\alpha}}\right)\langle\!\langle \phi(p_1)\cdots\phi(p_n)\rangle\!\rangle_{S^*}\\
&= \left(\frac{D-2}{2}+\gamma\right)\sum_{i=1}^n \frac{\partial}{\partial p_{i\alpha}}\langle\!\langle \phi(p_1)\cdots\phi(p_n)\rangle\!\rangle_{S^*}
\end{aligned} \tag{65}$$

where the right-hand side is determined by the full scale dimension of the scalar field at the fixed point.

If (58) does not hold, the left-hand side of (64) becomes $K_\alpha(0)$, and we obtain

$$\begin{aligned}&\sum_{i=1}^n \left(p_{i\nu} \frac{\partial^2}{\partial p_{i\alpha} \partial p_{i\nu}} - \frac{1}{2} p_{i\alpha} \frac{\partial^2}{\partial p_{i\beta}\partial p_{i\beta}} + D \frac{\partial}{\partial p_{i\alpha}} \right) \langle\!\langle \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_{S^*} \\ &= \left(\frac{D-2}{2} + \gamma \right) \sum_{i=1}^n \frac{\partial}{\partial p_{i\alpha}} \langle\!\langle \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_{S^*} - \langle\!\langle K_\alpha(0)\, \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_{S^*} \end{aligned} \tag{66}$$

instead.

VII. THE GAUSSIAN FIXED POINT

In this section we would like to give a concrete, though very simple, example of constructing the symmetric energy-momentum tensor using (38).

We consider the Wilson action at the gaussian fixed point, corresponding to the free massless theory with $\gamma = 0$:

$$S_G[\phi] = -\frac{1}{2} \int_p \frac{p^2}{K(p)} \phi(p)\phi(-p) \tag{67}$$

We then obtain

$$[\phi(p)] \equiv \frac{1}{K(p)} \left(\phi(p) + \frac{K(p)\,(1-K(p))}{p^2} \frac{\delta S_G}{\delta \phi(-p)} \right) = \phi(p)\,, \tag{68}$$

$$\begin{aligned}\mathcal{N}(p) &\equiv - \int_q K(q) \mathrm{e}^{-S_G} \frac{\delta}{\delta \phi(q)} \left(\phi(p+q) \mathrm{e}^{S_G} \right) \\ &= \int_{p_1,p_2} \delta(p_1+p_2-p)\, \frac{1}{2} \phi(p_1)\phi(p_2) \left(p_1^2 + p_2^2 \right)\end{aligned} \tag{69}$$

where a field independent constant proportional to $\delta(p)$ has been ignored for $\mathcal{N}(p)$. (Composite operators at S_G have been discussed extensively in Appendix of [4].) (38) gives

$$\begin{aligned}p_\mu \Theta_{\mu\nu}(p) &= \int_q K(q) \mathrm{e}^{-S_G} \frac{\delta}{\delta\phi(q)} \left((p+q)_\nu \phi(p+q) \mathrm{e}^{S_G} \right) \\ &= - \int_q (p+q)_\nu \phi(p+q) \phi(-q)\end{aligned} \tag{70}$$

Now, $\Theta_{\mu\nu}(p)$ is a symmetric tensor with scale dimension 0, and it can be written as

$$\Theta_{\mu\nu}(p) = \int_{p_1,p_2} \delta(p_1+p_2-p) \frac{1}{2} \phi(p_1)\phi(p_2)\, C_{\mu\nu}(p_1,p_2) \tag{71}$$

where

$$C_{\mu\nu}(p_1,p_2) = A\left(p_{1\mu}p_{1\nu} + p_{2\mu}p_{2\nu}\right) + B\left(p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu}\right) + \delta_{\mu\nu}\left(C\left(p_1^2+p_2^2\right) + D p_1 p_2 \right) \tag{72}$$

Substituting this into (70), we can determine B, C, D in terms of A as

$$C_{\mu\nu}(p_1, p_2) = \delta_{\mu\nu} p_1 p_2 - p_{1\mu} p_{2\nu} - p_{1\nu} p_{2\mu} + A \left\{ (p_1 + p_2)_\mu (p_1 + p_2)_\nu - (p_1 + p_2)^2 \delta_{\mu\nu} \right\} \tag{73}$$

where A is left arbitrary. Hence, we obtain the energy-momentum tensor

$$\Theta_{\mu\nu}(p) = \delta_{\mu\nu} \left[\frac{1}{2} \frac{1}{i} \partial_\alpha \phi \frac{1}{i} \partial_\alpha \phi \right](p) - \left[\frac{1}{i} \partial_\mu \phi \frac{1}{i} \partial_\nu \phi \right](p) + A \left(p_\mu p_\nu - p^2 \delta_{\mu\nu} \right) \left[\frac{1}{2} \phi^2 \right](p) \tag{74}$$

where

$$\left[\frac{1}{i} \partial_\mu \phi \frac{1}{i} \partial_\nu \phi \right](p) \equiv \int_{p_1, p_2} \delta(p_1 + p_2 - p) \phi(p_1) \phi(p_2)\, p_{1\mu} p_{2\nu}\,, \tag{75}$$

$$\left[\frac{1}{2} \phi^2 \right](p) \equiv \int_{p_1, p_2} \delta(p_1 + p_2 - p)\, \frac{1}{2} \phi(p_1) \phi(p_2) \tag{76}$$

The term proportional to A corresponds to

$$Y_{\mu\alpha,\nu\beta}(p) = A \left(\delta_{\mu\beta} \delta_{\nu\alpha} - \delta_{\mu\nu} \delta_{\alpha\beta} \right) \left[\frac{1}{2} \phi^2 \right](p) \tag{77}$$

which is a composite operator of scale dimension -2. Since the trace is given by

$$\Theta(p) = (D-2) \left[\frac{1}{2} \frac{1}{i} \partial_\alpha \phi \frac{1}{i} \partial_\alpha \phi \right](p) + A(1-D) p^2 \left[\frac{1}{2} \phi^2 \right](p) \tag{78}$$

and the equation-of-motion composite operator by

$$\mathcal{N}(p) = p^2 \left[\frac{1}{2} \phi^2 \right](p) - 2 \left[\frac{1}{2} \frac{1}{i} \partial_\alpha \phi \frac{1}{i} \partial_\alpha \phi \right](p) \tag{79}$$

we obtain

$$\begin{aligned} \mathcal{O}(p) &= \Theta(p) + \frac{D-2}{2} \mathcal{N}(p) \\ &= \left(A(1-D) + \frac{D-2}{2} \right) p^2 \left[\frac{1}{2} \phi^2 \right](p) \end{aligned} \tag{80}$$

which is quadratic in p. Hence, as is well known, the free massless theory has conformal invariance. By choosing

$$A = \frac{D-2}{2(D-1)} \tag{81}$$

we obtain an improved energy-momentum tensor for which $\mathcal{O}(p)$ vanishes identically.[3]

In Appendix C we consider an infinitesimal neighborhood of S_G and construct the energy-momentum tensor there.

VIII. CONCLUSIONS

In this paper we have considered how to construct the energy-momentum tensor $\Theta_{\mu\nu}(p)$, given a Wilson action which is invariant under translations and rotations. To make our task manageable we have introduced certain restrictions on the kind of Wilson actions we consider. We have assumed the continuum description of a Wilson action which is defined for an arbitrary continuous scalar field in D-dimensional space. In particular we have assumed that the ultraviolet cutoff of the theory is provided by a smooth cutoff function $K(p)$ of squared momentum p^2 which is itself rotation invariant.

Considering how long we have been familiar with the idea of Wilson actions, it is somewhat surprising that a problem as fundamental as construction of the energy-momentum tensor for a given Wilson action has never been considered fully before. It may be also a little surprising but reassuring that the naive translation invariance (13) and rotation invariance (19) of a Wilson action give us enough to construct the energy-momentum tensor $\Theta_{\mu\nu}(p)$ with expected properties, simply by following the existing formalism. In particular, we have derived the Ward identities (38) and (39), which amount to

$$\begin{aligned}
&\langle\!\langle p_\mu \Theta_{\mu\nu}(p)\,\phi(p_1)\cdots\phi(p_n)\rangle\!\rangle_S \\
&\quad = -\sum_{i=1}^{n}(p_i+p)_\nu\,\langle\!\langle \phi(p_1)\cdots\phi(p_i+p)\cdots\phi(p_n)\rangle\!\rangle_S
\end{aligned} \tag{82}$$

$$\begin{aligned}
&\left\langle\!\!\left\langle p_\nu\left(\frac{\partial\Theta_{\nu\mu}(p)}{\partial p_\alpha}-\frac{\partial\Theta_{\nu\alpha}(p)}{\partial p_\mu}\right)\phi(p_1)\cdots\phi(p_n)\right\rangle\!\!\right\rangle_S \\
&\quad = \sum_{i=1}^{n}\left\langle\!\!\left\langle \phi(p_1)\cdots\left((p+p_i)_\alpha\frac{\partial}{\partial p_\mu}-(p+p_i)_\mu\frac{\partial}{\partial p_\alpha}\right)\phi(p+p_i)\cdots\phi(p_n)\right\rangle\!\!\right\rangle_S
\end{aligned} \tag{83}$$

for the correlation functions. In demonstrating the existence of such $\Theta_{\mu\nu}(p)$, all we need is the assumption (1) that a local composite operator that vanishes at zero momentum is a spatial derivative.

We have shown that the symmetry $\Theta_{\mu\nu}(p)=\Theta_{\nu\mu}(p)$ and the translation invariance (38) determine $\Theta_{\mu\nu}(p)$ implicitly but almost uniquely. An additive ambiguity of the form (40) exists no matter what formalism we use.

Though we have considered only scalar theories in this paper, it should be straightforward to generalize our construction of the energy-momentum tensor to theories with spinor fields. For gauge theories, especially YM theories, we may need extra work to incorporate gauge invariance of the energy-momentum tensor.

Appendix A: Equation-of-motion Composite Operators

In this appendix we summarize, for the reader's convenience, the salient features of the equation-of-motion composite operators, which were originally called redundant operators in [13] in the context of the renormalization group. (The recent review article [14] adopts this original nomenclature.) More details can be found in §4 of [6].

Given a composite operator $\mathcal{O}(p)$, we define the corresponding equation-of-motion composite operator by

$$\mathcal{E}_{\mathcal{O}}(p) \equiv -\int_q K(q) \mathrm{e}^{-S[\phi]} \frac{\delta}{\delta \phi(q)} \left(\mathcal{O}(q+p) \mathrm{e}^{S[\phi]} \right) \tag{A1}$$

This has the following modified correlation functions:

$$\begin{aligned} \langle\!\langle \mathcal{E}_{\mathcal{O}}(p)\, \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_S &\equiv \prod_{i=1}^{n} \frac{1}{K(p_i)} \cdot \langle \mathcal{E}_{\mathcal{O}}(p)\, \phi(p_1) \cdots \phi(p_n) \rangle_S \\ &= \sum_{i=1}^{n} \langle\!\langle \phi(p_1) \cdots \mathcal{O}(p_i + p) \cdots \phi(p_n) \rangle\!\rangle_S \end{aligned} \tag{A2}$$

For example, if we choose $\mathcal{O}(p) = [\phi(p)]$, we obtain

$$\mathcal{E}_{[\phi]}(p) = \mathcal{N}(p) \equiv -\int_q K(q) \mathrm{e}^{-S[\phi]} \frac{\delta}{\delta \phi(q)} \left([\phi(q+p)]\ \mathrm{e}^{S[\phi]} \right) \tag{A3}$$

satisfying

$$\langle\!\langle \mathcal{N}(p)\, \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_S = \sum_{i=1}^{n} \langle\!\langle \phi(p_1) \cdots \phi(p_i + p) \cdots \phi(p_n) \rangle\!\rangle_S \tag{A4}$$

Especially, $\mathcal{N} \equiv \mathcal{N}(0)$ counts the number of fields:

$$\langle\!\langle \mathcal{N}\, \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_S = n\, \langle\!\langle \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_S \tag{A5}$$

We obtain another example by choosing

$$\mathcal{O}(p) = p_\mu \frac{\partial}{\partial p_\mu} [\phi(p)] \tag{A6}$$

We then obtain

$$\langle\!\langle \mathcal{E}_{\mathcal{O}}(p)\, \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_S = \sum_{i=1}^{n} \left\langle\!\!\left\langle \phi(p_1) \cdots (p+p_i)_\mu \frac{\partial \phi(p+p_i)}{\partial p_\mu} \cdots \phi(p_n) \right\rangle\!\!\right\rangle_S \tag{A7}$$

Especially, for $p = 0$, we obtain

$$\langle\!\langle \mathcal{E}_{\mathcal{O}}(0)\, \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_S = \sum_{i=1}^{n} \left\langle\!\!\left\langle \phi(p_1) \cdots p_{i\mu} \frac{\partial \phi(p_i)}{\partial p_{i\mu}} \cdots \phi(p_n) \right\rangle\!\!\right\rangle_S \tag{A8}$$

Appendix B: ERG Differential Equations

In [11] it was shown that (6) is equivalent to the ERG differential equation satisfied by $S_t \equiv R_t S$:

$$\partial_t \mathrm{e}^{S_t[\phi]} = \int_p \left[\left(\frac{\Delta(p)}{K(p)} + \frac{D+2}{2} - \gamma + p_\mu \frac{\partial}{\partial p_\mu} \right) \phi(p) \frac{\delta}{\delta \phi(p)} \right. \\ \left. + \frac{1}{p^2} \left\{ \Delta(p) - 2\gamma K(p) \left(1 - K(p)\right) \right\} \frac{1}{2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right] \mathrm{e}^{S_t[\phi]} \tag{B1}$$

where

$$\Delta(p) \equiv -2p^2 \frac{d}{dp^2} K(p) \tag{B2}$$

We wish to rewrite this equation as an operator equation using equation-of-motion composite operators.

We first rewrite (6) as

$$\langle\!\langle \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_{S_t} = \mathrm{e}^{t \cdot n \left(-\frac{D+2}{2} + \gamma\right)} \left\langle\!\!\left\langle \phi(p_1 \mathrm{e}^{-t}) \cdots \phi(p_n \mathrm{e}^{-t}) \right\rangle\!\!\right\rangle_S \tag{B3}$$

We then differentiate the above with respect to t to obtain

$$\langle\!\langle \partial_t S_t \, \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_{S_t} = n \left(-\frac{D+2}{2} + \gamma \right) \langle\!\langle \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_{S_t} \\ - \sum_{i=1}^n \left\langle\!\!\left\langle \phi(p_1) \cdots p_{i,\mu} \frac{\partial \phi(p_i)}{\partial p_{i,\mu}} \cdots \phi(p_n) \right\rangle\!\!\right\rangle_{S_t} \tag{B4}$$

Hence, using the two examples given in Appendix A, we obtain an operator equation

$$\partial_t S_t[\phi] = \left(-\frac{D+2}{2} + \gamma \right) \mathcal{N} + \int_p K(p) \mathrm{e}^{-S_t[\phi]} \frac{\delta}{\delta \phi(p)} \left(p_\mu \frac{\partial \left[\phi(p)\right]}{\partial p_\mu} \mathrm{e}^{S_t[\phi]} \right) \tag{B5}$$

This is equivalent to (B1). Hence, the Wilson action changes by an equation-of-motion operator under ERG.

The change of a composite operator under ERG is also an equation-of-motion operator. Let $\mathcal{O}(p)$ be a generic composite operator. Its ERG transformation is given by (8):

$$\left\langle\!\!\left\langle (R_t \mathcal{O})(p \mathrm{e}^t) \, \phi(p_1) \cdots \phi(p_n) \right\rangle\!\!\right\rangle_{S_t} = \mathrm{e}^{t \cdot n \left(-\frac{D+2}{2} + \gamma\right)} \left\langle\!\!\left\langle \mathcal{O}(p) \, \phi(p_1 \mathrm{e}^{-t}) \cdots \phi(p_n \mathrm{e}^{-t}) \right\rangle\!\!\right\rangle_S \tag{B6}$$

Differentiating this with respect to t, we obtain

$$\left\langle\!\!\left\langle \mathrm{e}^{-S_t} \partial_t \left((R_t \mathcal{O})(p \mathrm{e}^t) \, \mathrm{e}^{S_t} \right) \, \phi(p_1) \cdots \phi(p_n) \right\rangle\!\!\right\rangle_{S_t} \\ = n \left(-\frac{D+2}{2} + \gamma \right) \left\langle\!\!\left\langle (R_t \mathcal{O})(p \mathrm{e}^t) \, \phi(p_1) \cdots \phi(p_n) \right\rangle\!\!\right\rangle_{S_t} \\ - \sum_{i=1}^n \left\langle\!\!\left\langle (R_t \mathcal{O})(p \mathrm{e}^t) \, \phi(p_1) \cdots p_{i,\mu} \frac{\partial \phi(p_i)}{\partial p_{i,\mu}} \cdots \phi(p_n) \right\rangle\!\!\right\rangle_{S_t} \tag{B7}$$

This amounts to the operator equation

$$\begin{aligned}
&\mathrm{e}^{-S_t}\partial_t\left((R_t\mathcal{O})(p\mathrm{e}^t)\,\mathrm{e}^{S_t}\right)\\
&=\int_q K(q)\mathrm{e}^{-S_t[\phi]}\frac{\delta}{\delta\phi(q)}\left\{\left(\frac{D+2}{2}-\gamma+q_\mu\frac{\partial}{\partial q_\mu}\right)\left[(R_t\mathcal{O})(p\mathrm{e}^t)\,\phi(q)\right]\,\mathrm{e}^{S_t[\phi]}\right\}
\end{aligned}\tag{B8}$$

where $[\mathcal{O}'\,\phi(p)]$ is a composite operator defined by

$$[\mathcal{O}'\,\phi(p)]\equiv\mathcal{O}'\,\left[\phi(p)\right]+\frac{1-K(p)}{p^2}\frac{\delta\mathcal{O}'}{\delta\phi(-p)}\tag{B9}$$

By definition, we find

$$\langle\!\langle[\mathcal{O}'\phi(p)]\,\phi(p_1)\cdots\phi(p_n)\rangle\!\rangle_{S_t}=\langle\!\langle\mathcal{O}'\,\phi(p)\phi(p_1)\cdots\phi(p_n)\rangle\!\rangle_{S_t}\tag{B10}$$

The right-hand side of (B8) is an equation-of-motion composite operator.

At a fixed point S^*, a composite operator $\mathcal{O}$ with scale dimension $-y$ satisfies

$$(R_t\mathcal{O})(p\mathrm{e}^t)=\mathrm{e}^{yt}\mathcal{O}(p\mathrm{e}^t)\tag{B11}$$

Substituting this into (B8), we obtain

$$\left(y+p_\mu\frac{\partial}{\partial p_\mu}\right)\mathcal{O}(p)=\int_q K(q)\mathrm{e}^{-S^*}\frac{\delta}{\delta\phi(q)}\left\{\left(\frac{D+2}{2}-\gamma+q_\mu\frac{\partial}{\partial q_\mu}\right)[\mathcal{O}(p)\phi(q)]\,\mathrm{e}^{S^*}\right\}\tag{B12}$$

Appendix C: $\Theta_{\mu\nu}(p)$ in the Infinitesimal Neighborhood of S_G

We consider an infinitesimal neighborhood of the gaussian fixed point S_G by adding an arbitrary potential term with infinitesimal coefficients:

$$S\left\{g\right\}[\phi]=S_G[\phi]-\sum_{n=1}^{\infty}g_n\left[\frac{1}{(2n)!}\phi^{2n}\right](0)\tag{C1}$$

where

$$\begin{aligned}
\left[\frac{1}{(2n)!}\phi^{2n}\right](p)\equiv\sum_{k=0}^{n-1}\frac{C^k}{k!}\int_{p_1,\cdots,p_{2(n-k)}}&\delta(p_1+\cdots+p_{2(n-k)}-p)\\
&\times\frac{1}{(2(n-k))!}\,\phi(p_1)\cdots\phi(p_{2(n-k)})
\end{aligned}\tag{C2}$$

is a composite operator of scale dimension $-y_n\equiv n(D-2)-D$ defined at S_G. The constant C is defined by

$$C\equiv-\frac{1}{2}\int_q\frac{K(q)}{q^2}\tag{C3}$$

so that $[\phi^{2n}]$ is normal ordered. We find

$$\left\langle\!\!\left\langle \left[\phi^{2n}\right](pe^t)\,\phi(p_1 e^t)\cdots\phi(p_k e^t)\right\rangle\!\!\right\rangle_{S_G} = e^{t\left(-y_n - n\frac{D+2}{2}\right)} \left\langle\!\!\left\langle \left[\phi^{2n}\right](p)\,\phi(p_1)\cdots\phi(p_k)\right\rangle\!\!\right\rangle_{S_G} \tag{C4}$$

The symmetric energy-momentum tensor can be constructed from (38). We only give results here. To first order in g's, the energy-momentum tensor is given by

$$\begin{aligned}\Theta_{\mu\nu}(p) = {} & \delta_{\mu\nu}\left[\frac{1}{2}\frac{1}{i}\partial_\alpha\phi\frac{1}{i}\partial_\alpha\phi\right](p) - \left[\frac{1}{i}\partial_\mu\phi\frac{1}{i}\partial_\nu\phi\right](p) \\ & + A\left(p_\mu p_\nu - p^2\delta_{\mu\nu}\right)\left[\frac{1}{2}\phi^2\right](p) - \delta_{\mu\nu}\sum_{n=1}^{\infty} g_n\left[\frac{1}{(2n)!}\phi^{2n}\right](p)\end{aligned} \tag{C5}$$

where the composite operators are corrected to first order in g's as

$$\begin{aligned}\left[\frac{1}{i}\partial_\mu\phi\frac{1}{i}\partial_\nu\phi\right](p) = {} & \int_{p_1,p_2}\delta(p_1+p_2-p)\phi(p_1)\phi(p_2)p_{1\mu}p_{2\nu} - \tilde{J}_{\mu\nu}(p)\sum_{n=2}^{\infty}\tilde{g}_n\int_{p_1,\cdots,p_{2(n-1)}} \\ & \times\delta(p_1+\cdots+p_{2(n-1)}-p)\frac{1}{(2(n-1))!}\phi(p_1)\cdots\phi(p_{2(n-1)})\end{aligned} \tag{C6}$$

$$\begin{aligned}\left[\frac{1}{2}\phi^2\right](p) = {} & \int_{p_1,p_2}\delta(p_1+p_2-p)\frac{1}{2}\phi(p_1)\phi(p_2) - \tilde{I}(p)\sum_{n=2}^{\infty}\tilde{g}_n\int_{p_1,\cdots,p_{2(n-1)}} \\ & \times\delta(p_1+\cdots+p_{2(n-1)}-p)\frac{1}{(2(n-1))!}\phi(p_1)\cdots\phi(p_{2(n-1)})\end{aligned} \tag{C7}$$

The parameter $\tilde{g}_n$ is defined by

$$\tilde{g}_n \equiv \sum_{k=0}^{\infty} g_{n+k}\frac{C^k}{k!} \tag{C8}$$

The functions $\tilde{I}(p)$ and $\tilde{J}_{\mu\nu}(p)$ are well defined for $2 < D < 4$, and given by

$$\tilde{I}(p) \equiv \frac{1}{2}\int_q \frac{1-K(q)}{q^2}\frac{1-K(q+p)}{(q+p)^2} \tag{C9}$$

$$\begin{aligned}\tilde{J}_{\mu\nu}(p) \equiv {} & \int_0^\infty dt\, e^{t(D-2)}\left(2\int_q\frac{\Delta(q)}{q^2}\frac{1-K(q+pe^{-t})}{(q+pe^{-t})^2}(q+pe^{-t})_\mu(-q)_\nu\right. \\ & \left.+\frac{2}{D}\delta_{\mu\nu}\int_q\frac{\Delta(q)\,(1-K(q))}{q^2}\right) + \frac{2}{D(D-2)}\delta_{\mu\nu}\int_q\frac{\Delta(q)\,(1-K(q))}{q^2}\end{aligned} \tag{C10}$$

where $\Delta(p)$ is defined by (B2).

The trace is given, again to first order, by

$$\Theta(p) = -\frac{D-2}{2}\mathcal{N}(p) - \sum_{n=1}^{\infty} y_n g_n\left[\frac{1}{(2n)!}\phi^{2n}\right](p) \tag{C11}$$

if $A = (D-2)/(2(D-1))$ is chosen. Note that $y_n g_n$ is the beta function of g_n under ERG.

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[1] F. J. Belinfante, Physica **7**, 449 (1940)

[2] L. Rosenfeld, Mém. Acad. roy. Belgique **18**, 2 (1940)

[3] C. G. Callan Jr., S. R. Coleman, and R. Jackiw, Annals Phys. **59**, 42 (1970)

[4] K. Wilson and J. B. Kogut, Phys. Rept. **12**, 75 (1974)

[5] C. Becchi(1996), arXiv:hep-th/9607188 [hep-th]

[6] Y. Igarashi, K. Itoh, and H. Sonoda, Prog. Theor. Phys. Suppl. **181**, 1 (2010), arXiv:0909.0327 [hep-th]

[7] J. Polchinski, Nucl. Phys. **B303**, 226 (1988)

[8] B. Delamotte, M. Tissier, and N. Wschebor(2015), arXiv:1501.01776 [cond-mat.stat-mech]

[9] J. Wess, Nuovo Cimento **18**, 1086 (1960)

[10] O. J. Rosten(2014), arXiv:1411.2603 [hep-th]

[11] H. Sonoda(2015), arXiv:1503.08578 [hep-th]

[12] Y. Nakayama, Phys.Rept. **569**, 1 (2015), arXiv:1302.0884 [hep-th]

[13] F. J. Wegner, J. Phys. C **7**, 2098 (1974)

[14] O. J. Rosten, Phys. Rept. **511**, 177 (2012), arXiv:1003.1366 [hep-th]